An isoperimetric inequality for conjugation-invariant sets in the symmetric group

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Abstract

We prove an isoperimetric inequality for conjugation-invariant sets of size k in S_n, showing that these necessarily have edge-boundary considerably larger than some other sets of size k (provided k is small). Specifically, let T_n denote the Cayley graph on S_n generated by the set of all transpositions. We show that if $A \subset S_n$ is a conjugation-invariant set with $|A| = pn! \leq n!/2$, then the edge-boundary of A in T_n has size at least

$$c \cdot \frac{\log_2\left(\frac{1}{p}\right)}{\log_2 \log_2\left(\frac{2}{p}\right)} \cdot n \cdot |A|,$$

where c is an absolute constant. (This is sharp up to an absolute constant factor, when $p = \Theta(1/s!)$ for any $s \in \{1, 2, \ldots, n\}$.) It follows that if $p = n^{-\Theta(1)}$, then the edge-boundary of a conjugation-invariant set of measure p is necessarily a factor of $\Omega(\log n / \log\log n)$ larger than the minimum edge-boundary over all sets of measure p.

1 Introduction

Isoperimetric problems are classical objects of study in mathematics. In general, they ask for the smallest possible ‘boundary’ of a set of a given ‘size’. For example, of all shapes in the plane with area 1, which has the smallest perimeter? The ancient Greeks were sure that the answer is a circle, but it was not until the 19th century (with the work of Weierstrass) that this was proved rigorously.

In the last fifty years, ‘discrete’ isoperimetric problems have been extensively studied. These deal with the boundaries of sets of vertices in graphs. Here, there are two different notions of boundary. If $G = (V, E)$ is a graph, and $A \subset V$, the *vertex-boundary of A in G* is the set of all vertices in $V \setminus A$ which have a

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neighbour in A. (This is sometimes denoted by $b_G(A)$.) Similarly, the *edge-boundary of A in G* is the set of all edges of G between A and $V \setminus A$. (This is often denoted by $\partial_G(A)$.) The *vertex-isoperimetric problem for G* asks for the minimum possible size of the vertex-boundary of a k-element subset of V, for each $k \in \mathbb{N}$. Similarly, the *edge-isoperimetric problem for G* asks for the minimum possible size of the edge-boundary of a k-element subset of V, for each $k \in \mathbb{N}$.

A well-known example arises from taking the graph G to be the n-dimensional hypercube Q_n, the graph with vertex-set $\{0,1\}^n$, where x and y are joined by an edge if and only if they differ in exactly one coordinate. It turns out that for any $k \in \{1,2,\ldots,2^n\}$, the edge-boundary of a k-element subset of $\{0,1\}^n$ is minimized by taking the first k elements of the binary ordering on $\{0,1\}^n$. (This was proved by Harper [9], Lindsey [14], Bernstein [5], and Hart [10].) It follows that if $A \subset \{0,1\}^n$ with $|A| = 2^{n-t}$, where $t \in \{1,2,\ldots,n\}$ then $|\partial_{Q_n}(A)| \geq t2^{n-t}$. Equality holds if and only if A is a subcube of codimension t.

The reader is referred to [13] for a survey of results and open problems in the field of discrete isoperimetric inequalities.

It is natural to ask what happens to the minimum size of the edge-boundary if one imposes some kind of symmetry constraint on the set A. For example, we say that a set $A \subset \{0,1\}^n$ is *transitive-symmetric* if there exists a transitive subgroup $H \leq S_n$ such that $\sigma(A) = A$ for all $\sigma \in H$. (Here, $\sigma(A) := \{\sigma(x) : x \in A\}$, where $\sigma(x)$ is defined by $(\sigma(x))_i = x_{\sigma^{-1}(i)}$ for each $i \in [n]$.) In other words, A is transitive-symmetric if there exists a group which acts transitively on the coordinates and leaves A invariant. It turns out that a transitive-symmetric set in $\{0,1\}^n$ must have considerably larger edge-boundary than some other sets of the same size. This follows from the celebrated KKL theorem on influences. Recall that if $A \subset \{0,1\}^n$, and $i \in [n]$, the *influence* $\mathrm{Inf}_i(A)$ of the ith coordinate on A is defined by

$$\mathrm{Inf}_i(A) = \frac{|\{x \in \{0,1\}^n : \text{ exactly one of } x \text{ and } x^i \text{ is in } A\}|}{2^n},$$

where x^i denotes x with the ith coordinate flipped. Equivalently, if $E_i(Q_n)$ denotes the set of all 2^{n-1} direction-i edges of Q_n (meaning, edges of the form $\{x, x^i\}$), then

$$\mathrm{Inf}_i(A) = \frac{|\partial_{Q_n}(A) \cap E_i(Q_n)|}{|E_i(Q_n)|} = \frac{|\partial_{Q_n}(A) \cap E_i(Q_n)|}{2^{n-1}}.$$

Note that

$$|\partial_{Q_n}(A)| = 2^{n-1} \sum_{i=1}^{n} \mathrm{Inf}_i(A).$$

Kahn, Kalai and Linial [12] proved the following.

Theorem 1 (Kahn, Kalai, Linial). *Let $A \subset \{0,1\}^n$ with $|A| = p2^n$. Then provided n is larger than an absolute constant, there exists a coordinate $i \in [n]$ such that*

$$Inf_i(A) \geq p(1-p)\frac{\ln n}{n}.$$

If A is transitive-symmetric, then all its influences are the same, so by Theorem 1, its edge-boundary must satisfy

$$|\partial_{Q_n}(A)| \geq 2^{n-1}p(1-p)\ln n, \tag{1}$$

provided n is larger than an absolute constant. When $|A| = 2^{n-t}$ (so $p = 2^{-t}$), and $t \in \mathbb{N}$ is bounded, this is a factor of approximately $\ln n$ larger than the 'unrestricted' minimum edge-boundary of $t \cdot p2^n$, attained by a subcube. For $n^{-\Omega(1)} \leq p \leq 1 - n^{-\Omega(1)}$, the 'tribes' construction of Ben-Or and Linial [4] gives a transitive-symmetric family $A \subset \{0,1\}^n$ with $|A| = pn!$ and with

$$|\partial_{Q_n}(A)| = \Theta(2^{n-1}p(1-p)\ln n),$$

showing that (1) is sharp up to an absolute constant factor.

We study an analogue of this phenomenon for the symmetric group S_n, the group of all permutations of $\{1, 2, \ldots, n\}$. Let T_n denote the transposition graph on S_n. This is the Cayley graph on S_n generated by the set of all transpositions, i.e. the graph with vertex-set S_n, where two permutations $\sigma, \pi \in S_n$ are joined by an edge if and only if $\sigma\pi^{-1}$ is a transposition. In other words, writing a permutation $\sigma \in S_n$ in sequence notation $(\sigma(1), \sigma(2), \ldots, \sigma(n))$, two permutations are joined by an edge if and only if their sequences differ by swapping two elements. We are interested in the edge-boundary of sets in T_n. If $A \subset S_n$, we let $\partial A = \partial_{T_n}(A)$ denote the edge-boundary of A in T_n. We define the *lexicographic order* on S_n by $\sigma < \pi$ if and only if $\sigma(j) < \pi(j)$, where $j = \min\{i \in [n] : \sigma(i) \neq \pi(i)|\}$. Ben Efraim [3] made the following conjecture.

Conjecture 2 (Ben Efraim). *For any $\mathcal{A} \subset S_n$, $|\partial\mathcal{A}| \geq |\partial\mathcal{C}|$, where $\mathcal{C}$ denotes the initial segment of the lexicographic order on S_n of size $|\mathcal{A}|$.*

(Here, the *initial segment of size k* of the lexicographic order means the first k smallest elements of S_n in the lexicographic order.)

To date, Conjecture 2 is known only for sets of size $c(n-1)!$ where $c \in \mathbb{N}$ (see Corollary 6), and for sets of size $(n-t)!$, where n is sufficiently large depending on t (see [7]).

Note that for any $t \in [n]$, the initial segment of the lexicographic order of size $(n-t)!$ is precisely the set of all permutations in S_n fixing $[t]$ pointwise, which has edge-boundary of size $t(n-1)(n-t)!$. Similarly, it can be checked (see Appendix) that if $A \subset S_n$ is an initial segment of the lexicographic ordering on S_n with $(n-t-1)! < |A| \leq (n-t)!$ for some $t \in \{0, 1, 2, \ldots, n-1\}$, then

$$|\partial A| \leq (t + 3/2)(n-1)|A|. \tag{2}$$

In this paper, we study the edge-boundary of subsets of S_n which are conjugation-invariant, i.e. unions of conjugacy-classes of S_n. We feel that this is a natural invariance requirement to impose upon subsets of S_n. We prove the following.

Theorem 3. *There exists an absolute constant $c > 0$ such that the following holds. Let $A \subset S_n$ be a conjugation-invariant family of permutations with $0 < |A| \le n!/2$, and let $p = |A|/n!$ denote the measure of A. Then*

$$|\partial A| \ge c \cdot \frac{\log_2\left(\frac{1}{p}\right)}{\log_2\log_2\left(\frac{2}{p}\right)} \cdot n \cdot |A|.$$

Note that an analogous result follows immediately for sets of size greater than $n!/2$, by applying the above result to A^c, since $\partial(A^c) = \partial A$.

Comparing the bound in the above theorem with (2), we see that if $p = n^{-\Theta(1)}$, then the edge-boundary of a conjugation-invariant set of measure p is necessarily a factor of $\Omega(\log n/\log\log n)$ larger than the minimum edge-boundary over all sets of measure p.

Observe that Theorem 3 is sharp up to the value of the absolute constant c, for a large number of different values of p. Indeed, for each $s \in [n]$, let

$$A_s = \{\sigma \in S_n : \ \sigma \text{ has at least } s \text{ fixed points}\}. \tag{3}$$

We make the following.

Claim 1. *For each $s \in [n-2]$,*

$$\frac{n!}{3s!} \le |A_s| \le \frac{n!}{s!}. \tag{4}$$

Proof of claim. Recall that a *derangement* of $[m]$ is a permutation of $[m]$ with no fixed point. Let d_m denotes the number of derangements of $[m]$. By the inclusion-exclusion formula, we have

$$d_m = \sum_{i=0}^{m}(-1)^i\binom{m}{i}(m-i)! = m!\sum_{i=0}^{m}(-1)^i\frac{1}{i!} \ge \frac{m!}{3} \quad \forall m \ge 2.$$

Note that $\binom{n}{s}d_{n-s}$ is precisely the number of permutations in S_n with exactly s fixed points. Hence, we have

$$\frac{n!}{3s!} = \tfrac{1}{3}\binom{n}{s}(n-s)! \le \binom{n}{s}d_{n-s} \le |A_s| \le \binom{n}{s}(n-s)! = \frac{n!}{s!} \quad \forall s \in [n-2],$$

proving the claim. □

Hence, if $p = p_s = |A_s|/n!$, then we have

$$\frac{1}{3s!} \le p \le \frac{1}{s!},$$

so

$$s = \Theta\left(\frac{\log_2(\frac{1}{p})}{\log_2\log_2(\frac{2}{p})}\right). \tag{5}$$

Note that (5) also holds in the case $s \in \{n-1, n\}$, where $A_s = \{\mathrm{Id}\}$.

Now observe that

$$|\partial(A_s)| \leq |A_s|(s+1)(n-1),$$

as an element $\sigma \in A_s$ is incident with at least one edge of $\partial(A_s)$ only if it has either s or $s+1$ fixed points, and then there are at most $(s+1)(n-1)$ transpositions τ such that $\sigma\tau \notin A_s$. Putting everything together, we have

$$|\partial(A_s)| = \Theta\left(\frac{\log_2(\frac{1}{p})}{\log_2\log_2(\frac{2}{p})}\right) \cdot n \cdot |A_s|,$$

confirming the sharpness of Theorem 3.

Note that

$$|A_1| = n! - d_n = n!\left(1 - \sum_{i=0}^{n}(-1)^i\frac{1}{i!}\right) = (1 - 1/e + o(1))n!,$$

and

$$|\partial A_1| \leq 2 \cdot (n-1) \cdot |A_1|,$$

which is within an absolute constant factor of the lower bound

$$|\partial A| \geq (1/e)(1 - 1/e + o(1)) \cdot n \cdot n!$$

given by plugging in $|A| = (1 - 1/e + o(1))n!$ into Corollary 6 (see later). So for sets of constant measure, imposing the condition of conjugation-invariance cannot increase the minimum possible edge-boundary by more than a constant factor.

It is natural to ask what happens when one imposes a weaker condition than conjugation-invariance. We say that $A \subset S_n$ is *transitive-conjugation-invariant* if there exists a transitive subgroup $H \leq S_n$ such that A is invariant under conjugation by any permutation in H — that is, for all $\sigma \in S_n$ and all $\pi \in H$, we have $\pi\sigma\pi^{-1} \in H$. However, it turns out that imposing this condition does not increase the minimum possible edge-boundary by more than an absolute constant factor, when $|A| = \Theta(\frac{n}{k}(n-k)!)$ for some $k \mid n$ (provided Conjecture 2 holds). To see this, let $n, k \in \mathbb{N}$ with $k \mid n$. For each $i \in [n/k]$, let $I_i = \{(i-1)k+1, (i-1)k+2, \ldots, \ldots, ik\}$. Let

$$A = \{\sigma \in S_n : \ \sigma \text{ fixes some } I_i \text{ pointwise}\}.$$

Clearly, A is transitive-conjugation-invariant; we may take the group H to be the group of all permutations preserving the partition $I_1 \cup I_2 \cup \ldots \cup I_{n/k}$. In the case $k = 1$, we have $A = A_1$ (as defined above), and in the case $k = n$, we have $A = A_n = \{\mathrm{Id}\}$. Hence, we may assume that $1 < k \leq n/2$. We have

$$\frac{n}{2k}(n-k)! < \frac{n}{k}(n-k)! - \binom{n/k}{2}(n-2k)! \leq |A| \leq \frac{n}{k}(n-k)!,$$

using the Bonferroni inequalities, so

$$(n-k)! < |A| \leq (n-k+1)!.$$

On the other hand, since a permutation fixing I_i pointwise has at most $k(n-1)$ neighbours which do not fix I_i pointwise, we have

$$|\partial A| \leq k(n-1)|A|.$$

This is within an absolute constant factor of the bound (2) when $t = k - 1$.

Our method of proving Theorem 3 is algebraic. We use the well-known expression

$$|\partial A| = 1_A^\top L 1_A, \tag{6}$$

where L denotes the Laplacian of T_n, and 1_A denotes the indicator function of the set $A \subset S_n$. (Of course, this holds when T_n is replaced by any finite graph G, and L is the Laplacian of G, for any subset $A \subset V(G)$.) We consider the expansion of the right-hand side of (6) in terms of the eigenvalues of L and the L^2-weights of 1_A on each eigenspace of L. We use known results to analyse the eigenvalues of L. Most of the work of our proof is in showing that if A is conjugation-invariant, then most of the L^2-weight of 1_A is on eigenspaces of L corresponding to 'large' eigenvalues. (Here, the meaning of 'large' depends on the size of the set A.) To do this, we use tools from the representation theory of the symmetric group.

Notation and background

Before proving Theorem 3, we first describe some notation and background.

Notation

Throughout, we will write $\log(t)$ for $\log_2(t)$, and $\ln(t)$ for $\log_e(t)$. As usual, if $n \in \mathbb{N}$, we write $[n]$ for the set $\{1, 2, \ldots, n\}$. If X is a set, and $f, g : X \to \mathbb{R}$ are functions, we write $g = O(f)$ (and $f = \Omega(g)$) if there exists an absolute constant $C > 0$ such that $|g(x)| \leq C|f(x)|$ for all $x \in X$. We write $g = \Theta(f)$ if $g = O(f)$ and $g = \Omega(f)$ both hold.

Background

If $G = (V, E)$ is a finite graph, we define its *Laplacian* $L = L_G$ to be the matrix with rows and columns indexed by V, where

$$L_{u,v} = \begin{cases} d(v) & \text{if } u = v; \\ -1 & \text{if } u \neq v,\ \{u,v\} \in E(G); \\ 0 & \text{if } u \neq v,\ \{u,v\} \notin E(G). \end{cases}$$

For any $x \in \mathbb{R}^V$, we have

$$x^\top L x = \sum_{\{u,v\} \in E(G)} (x(u) - x(v))^2, \tag{7}$$

so L is a positive semidefinite matrix. Note that the constant vector $(1, 1, \ldots, 1) \in \mathbb{R}^V$ is always an eigenvector of L with eigenvalue 0. If G is a connected graph, then L has eigenvalue 0 with multiplicity 1. We write $\mu_2 = \mu_2(L)$ for the second-smallest eigenvalue of L. The following well-known theorem supplies a lower bound for the edge-boundary of a set $A \subset V(G)$, in terms of μ_2.

Theorem 4 (Alon, Milman [1])**.** *Let G be a connected graph. If $A \subset V(G)$, then*

$$|\partial_G(A)| \geq \mu_2 \frac{|A|(|V| - |A|)}{|V|}.$$

We give the standard proof (due to Alon and Milman), as we will need to refer to it later.

Proof. Let $1_A \in \{0,1\}^V$ denote the characteristic vector of A, defined by

$$1_A(v) = \begin{cases} 1 & \text{if } v \in A; \\ 0 & \text{if } v \notin A. \end{cases}$$

We equip $\mathbb{R}^V$ with the inner product

$$\langle x, y \rangle = \frac{1}{|V|} \sum_{v \in V} x(v) y(v).$$

Let $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \ldots \leq \mu_{|V|}$ denote the eigenvalues of L, repeated with their multiplicities, and let $w_1 = (1, \ldots, 1), w_2, \ldots, w_{|V|}$ be an orthonormal basis of $\mathbb{R}^V$ consisting of eigenvectors of L, such that w_i is a μ_i-eigenvector of L. Write

$$1_A = \sum_{i=1}^{|V|} b_i w_i$$

as a linear combination of the w_i. Then, by orthonormality, we have

$$|A|/|V| = \langle 1_A, 1_A \rangle = \sum_{i=1}^{|V|} b_i^2.$$

Moreover, we have

$$b_1 = \langle 1_A, (1, \ldots, 1) \rangle = |A|/|V|.$$

Using (7), we have

$$
\begin{aligned}
|\partial_G(A)| &= 1_A^\top L 1_A \\
&= |V|\langle 1_A, L1_A\rangle \qquad (8)\\
&= |V|\sum_{i=1}^{|V|}\mu_i b_i^2 \\
&\geq |V|\mu_2\sum_{i=2}^{|V|} b_i^2 \\
&= |V|\mu_2\left(\frac{|A|}{|V|}-\frac{|A|^2}{|V|^2}\right) \\
&= \mu_2\frac{|A|(|V|-|A|)}{|V|},
\end{aligned}
$$

proving Theorem 4. □

We also need some background on the representation theory of S_n. This can be found, for example, in [11].

Let $L_n = L_{T_n}$ denote the Laplacian matrix of the transposition graph T_n. We equip $\mathbb{R}^{S_n}$ with the inner product

$$\langle x, y\rangle = \frac{1}{n!}\sum_{\sigma\in S_n} x(\sigma)y(\sigma), \tag{9}$$

and we let

$$\|x\| = \sqrt{\frac{1}{n!}\sum_{\sigma\in S_n} x(\sigma)^2}$$

denote the corresponding L^2-norm. Note that in the sequel, we will pass freely between vectors in $\mathbb{R}^{S_n}$ and the corresponding functions from S_n to $\mathbb{R}$.

The eigenspaces of L_n (and the corresponding eigenvalues) were determined by Diaconis and Shahshahani [6]. They are in a natural one-to-one correspondence with the irreducible characters[1] of S_n over $\mathbb{R}$, and in fact each irreducible character (when viewed as a vector) lies in the corresponding eigenspace. In turn, the irreducible characters of S_n over $\mathbb{R}$ are in a natural one-to-one correspondence with the partitions of n. Recall the following.

Definition. *If $n \in \mathbb{N}$, a* partition *of n is a monotone non-increasing sequence of positive integers with sum n. In other words, $\alpha = (\alpha_1, \ldots, \alpha_l)$ is a partition of n if $\alpha_i \in \mathbb{N}$ for all $i \in [l]$, $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_l$, and $\sum_{i=1}^{l}\alpha_i = n$. For example, $(3, 2, 2)$ is a partition of 7. If α is a partition of n, then we sometimes write $\alpha \vdash n$. For each $n \in \mathbb{N}$, we write $p(n)$ for the number of partitions of n; for convenience, we define $p(0) = 1$.*

[1] Recall that if Γ is a finite group, an *irreducible character* of Γ is a character of an irreducible representation of Γ.

If $\alpha = (\alpha_1, \dots, \alpha_k)$ is a partition of n, we write χ_α for the corresponding irreducible character of S_n over $\mathbb{R}$, and we write μ_α for the corresponding eigenvalue of L_n. Diaconis and Shahshahani derived the following useful formula.

$$\mu_\alpha = \binom{n}{2} - \frac{1}{2}\sum_{i=1}^{k}\left[(\alpha_i - i)(\alpha_i - i + 1) - i(i-1)\right]. \tag{10}$$

To analyse these eigenvalues, it is useful to consider the *dominance ordering*, a partial order on the set of partitions of n which is defined as follows.

Definition. *If $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_l)$ are distinct partitions of n, we say that α is greater than β in the* dominance ordering *(and we write $\alpha \rhd \beta$) if $\sum_{i=1}^{r} \alpha_i \geq \sum_{i=1}^{r} \beta_i$ for all $r \in \mathbb{N}$. (Here, $\alpha_i := 0$ for all $i > k$, and similarly $\beta_i := 0$ for all $i > l$.)*

Diaconis and Shahshahani observed the following.

Lemma 5. *The eigenvalues $(\mu_\alpha)_{\alpha \vdash n}$ are monotonically non-increasing with respect to the dominance ordering on the set of partitions of n: if α and β are partitions of n with $\beta \trianglerighteq \alpha$, then $\mu_\beta \leq \mu_\alpha$.*

Notice that $\mu_{(n)} = 0$, $\mu_{(n-1,1)} = n$, and if $\alpha \neq (n)$, then $(n-1,1) \trianglerighteq \alpha$, so $\mu_\alpha \geq \mu_{(n-1,1)}$. It follows that $\mu_2(L_n) = n$. Plugging this into Theorem 4 yields the following, essentially due to Diaconis and Shahshahani.

Corollary 6. *If $A \subset S_n$, then*

$$|\partial A| \geq \frac{|A|(n! - |A|)}{(n-1)!}.$$

This verifies Conjecture 2 when $|A| = c(n-1)!$ for some $c \in \mathbb{N}$. (Note that equality holds in Corollary 6 when $A = \{\sigma \in S_n : \sigma(1) \in \{1, 2, \dots, c\}$.)

The following Corollary of Lemma 5 and (10) will be useful for us.

Corollary 7. *For any $t \in \{0, 1, 2, \dots, n\}$, if α is a partition of n with $\alpha_1 \leq n - t$, then we have $(n-t, t) \trianglerighteq \alpha$, so*

$$\mu_\alpha \geq \mu_{(n-t,t)} = tn - t^2 + t.$$

Next, we need a fact about conjugation-invariant functions.

Definition. *If $f : S_n \to \mathbb{R}$, we say f is a* class function *if it is conjugation-invariant, i.e.*

$$f(\pi\sigma\pi^{-1}) = f(\sigma) \quad \forall \sigma, \pi \in S_n.$$

Fact 1. *The irreducible characters of S_n over $\mathbb{R}$ are an orthonormal basis for the vector space of real-valued class functions on S_n, under the inner product (9).*

We now need some facts about *permutation characters*. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a partiton of n. The *Young diagram* of α is an array of n boxes, or 'cells', having k left-justified rows, where row i contains α_i cells. For example, the Young diagram of the partition $(3, 2, 2)$ is:

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} & \\ \cline{1-2}\end{array}$$

If the array contains the numbers $\{1, 2, \ldots, n\}$ inside the cells, we call it an *α-tableau*, or a *tableau of shape α*; for example,

$$\begin{array}{|c|c|c|}\hline 6 & 1 & 7 \\ \hline 5 & 4 \\ \cline{1-2} 3 & 2 \\ \cline{1-2}\end{array}$$

is a $(3, 2, 2)$-tableau. Two α-tableaux are said to be *row-equivalent* if they have the same numbers in each row.

An *α-tabloid* is an α-tableau with unordered row entries (or, more formally, a row-equivalence class of α-tableaux). For example, the $(3, 2, 2)$-tableau above corresponds to the following $(3, 2, 2)$-tabloid:

$$\begin{array}{cccccc} \{ & 1 & 6 & 7 & \} \\ \{ & 4 & 5 & & \} \\ \{ & 2 & 3 & & \} \end{array}$$

where each row is a set, not a sequence. Consider the natural left action of S_n on the set X^α of all α-tabloids. For example, the permutation $(1, 5)(2, 6, 4)(3)(7)$ (written in disjoint cycle notation) acts on the tabloid above as follows:

$$(1,5)(2,6,4)(3,7) \left(\begin{array}{ccccc} \{ & 1 & 6 & 7 & \} \\ \{ & 4 & 5 & & \} \\ \{ & 2 & 3 & & \} \end{array} \right) = \begin{array}{ccccc} \{ & 3 & 4 & 5 & \} \\ \{ & 1 & 2 & & \} \\ \{ & 6 & 7 & & \} \end{array}$$

Let $M^\alpha = \mathbb{R}[X^\alpha]$ be the corresponding permutation representation, i.e. the real vector space with basis X^α and S_n action given by extending linearly. We write ξ_α for the character of this representation. The $\{\xi_\alpha\}_{\alpha \vdash n}$ are called the *permutation characters* of S_n. If $\sigma \in S_n$, then $\xi_\alpha(\sigma)$ is simply the number of α-tabloids fixed by σ.

We can express the irreducible characters in terms of the permutation characters using the *determinantal formula*: for any partition α of n,

$$\chi_\alpha = \sum_{\pi \in S_n} \operatorname{sign}(\pi) \xi_{\alpha - \mathrm{id} + \pi}. \tag{11}$$

Here, if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$, then $\alpha - \mathrm{id} + \pi$ is defined to be the sequence

$$(\alpha_1 - 1 + \pi(1), \alpha_2 - 2 + \pi(2), \ldots, \alpha_l - l + \pi(l)).$$

If this sequence has all its entries non-negative, then we let $\overline{\alpha - \mathrm{id} + \pi}$ be the partition of n obtained by reordering its entries, and we define $\xi_{\alpha-\mathrm{id}+\pi} = \xi_{\overline{\alpha-\mathrm{id}+\pi}}$. If the sequence has a negative entry, then we define $\xi_{\alpha-\mathrm{id}+\pi} = 0$. It is easy to see that if ξ_β appears on the right-hand side of (11), then $\beta \trianglerighteq \alpha$, so the determinantal formula expresses χ_α in terms of $\{\xi_\beta : \ \beta \trianglerighteq \alpha\}$. We may rewrite the determinantal formula as

$$\chi_\alpha = \sum_{\beta \trianglerighteq \alpha} c_{\alpha\beta}\xi_\beta, \tag{12}$$

where $c_{\alpha\beta} \in \mathbb{Z}$ for each $\beta \trianglerighteq \alpha$.

We need the following.

Lemma 8. *Let $u \in \mathbb{N}$, and let α be a partition of n with $\alpha_1 = n - u$. Then*

$$\sum_{\beta \trianglerighteq \alpha} |c_{\alpha\beta}| \leq (u+1)!.$$

Proof. Fix an integer $i \geq u+2$. Note that $\alpha_i = 0$. If $\pi \in S_n$ with $\pi(i) < i$, then $\alpha(i) - i + \pi(i) < 0$, so $\xi_{\alpha-\mathrm{id}+\pi} = 0$. Hence, for any permutation $\pi \in S_n$ such that $\xi_{\alpha-\mathrm{id}+\pi} \neq 0$, we must have $\pi(i) \geq i$ for all $i \in \{u+2, u+3, \ldots, n\}$, so $\pi(i) = i$ for all $i \in \{u+2, u+3, \ldots, n\}$, i.e. $\pi \in S_{[u+1]}$. This proves the lemma. $\square$

Observe that if $u \leq n/2$, then the number of partitions α of n with $\alpha_1 = n-u$ is precisely $p(u)$. Hence, if $t \leq n/2+1$, then the number of partitions α of n with $\alpha_1 \geq n-t+1$ is precisely $\sum_{i=0}^{t-1} p(i)$. We will use the following crude bound on this quantity.

Lemma 9. *If $t \in \mathbb{N}$, then*

$$\sum_{i=0}^{t-1} p(i) \leq t!.$$

Proof. We have $p(i) \leq i!$ for all $i \in \mathbb{N} \cup \{0\}$, since $p(i)$ is the number of conjugacy-classes in the symmetric group S_i. Hence,

$$\sum_{i=0}^{t-1} p(i) \leq \sum_{i=0}^{t-1} i! \leq t!.$$

$\square$

2 Proof of Theorem 3

Let $A \subset S_n$ be a conjugation-invariant set with $0 < |A| = pn! \leq n!/2$. Note that by choosing $c > 0$ small enough, we may assume that $p \leq \epsilon_0$, for any absolute constant ϵ_0. Indeed, if $\epsilon_0 \leq p \leq 1/2$, then by Corollary 6, we have

$$|\partial A| \geq \tfrac{1}{2} n \cdot |A| \geq c \cdot \frac{\log\left(\frac{1}{\epsilon_0}\right)}{\log\log\left(\frac{2}{\epsilon_0}\right)} \cdot n \cdot |A| \geq c \cdot \frac{\log\left(\frac{1}{p}\right)}{\log\log\left(\frac{2}{p}\right)} \cdot n \cdot |A|,$$

provided c is chosen to be sufficiently small depending on ϵ_0.

From (8) applied to T_n, we have

$$|\partial A| = n!\,\langle\, 1_A, L_n 1_A \,\rangle\,. \tag{13}$$

Since the irreducible characters of S_n over $\mathbb{R}$ are an orthonormal basis for the space of real-valued class functions on S_n, we may write

$$1_A = \sum_{\alpha \vdash n} w_\alpha \chi_\alpha, \tag{14}$$

where $w_\alpha = \langle 1_A, \chi_\alpha \rangle \in \mathbb{R}$ for each $\alpha \vdash n$. Since χ_α is an eigenvector of L_n with eigenvalue μ_α, substituting this into (13) gives

$$|\partial A| = n! \sum_{\alpha \vdash n} \mu_\alpha w_\alpha^2. \tag{15}$$

Now define $K = K(p)$ by

$$K^{2K} = \frac{1}{p}. \tag{16}$$

We pause to note simple lower and upper bounds on K. Taking the logarithm of both sides of (16) gives:

$$2K \log K = \log \tfrac{1}{p} \leq \log \tfrac{2}{p}.$$

Thus $K < \log \frac{2}{p}$, which implies $\log K < \log\log \frac{2}{p}$. Therefore:

$$K = \frac{\log \frac{1}{p}}{2 \log K} \geq \frac{\log \frac{1}{p}}{2 \log\log \frac{2}{p}}.$$

On the other hand, since $A \neq \emptyset$, we have $p \geq 1/n!$, so

$$K^{2K} \leq n! \leq n^n,$$

and therefore $K \leq n$. Putting these two bounds together, we have

$$\frac{\log \frac{1}{p}}{2 \log\log \frac{2}{p}} \leq K \leq n. \tag{17}$$

Let M be a large, fixed integer. (For concreteness, we may take $M = 18$.) Define $t_p = \lfloor K/M \rfloor$; note that

$$\frac{\log \frac{1}{p}}{4M \log\log \frac{2}{p}} \leq t_p \leq \frac{n}{M}. \tag{18}$$

We need the following bound on $|w_\alpha|$ for $\alpha_1 > n - t_p$.

Proposition 10. *Let $\alpha \vdash n$ with $\alpha_1 = n - t$, where $t < t_p$. Then*

$$|w_\alpha| \le \frac{1}{K^{2K(1-\frac{8}{M})}}.$$

Proof. By (14) and the orthonormality of $\{\chi_\alpha\}_{\alpha \vdash n}$, we have

$$w_\alpha = \langle 1_A, \chi_\alpha \rangle = \frac{1}{n!} \sum_{\sigma \in A} \chi_\alpha(\sigma) \tag{19}$$

By (12), we have

$$|\chi_\alpha(\sigma)| \le \sum_{\beta \trianglerighteq \alpha} |c_{\alpha\beta}| \xi_\beta(\sigma). \tag{20}$$

For each tabloid T of shape $\beta = (\beta_1, \ldots, \beta_n)$ let $\tilde{T}$ be the tabloid of shape $\tilde{\beta} := (\beta_1, n - \beta_1)$ obtained by collapsing all the rows other than the first one into a single row of length $n - \beta_1 = \sum_{i=2}^n \beta_i$. We observe that for every permutation $\sigma \in S_n$, if σ fixes the tabloid T, then it also fixes the tabloid $\tilde{T}$. Moreover, the mapping $T \mapsto \tilde{T}$ is a surjection, and is at most $(n - \beta_1)!$ to 1. Recalling that $\xi_\beta(\sigma)$ is the number of β-tabloids fixed by σ, we obtain

$$\xi_\beta(\sigma) \le (n - \beta_1)! \cdot \xi_{\tilde{\beta}}(\sigma).$$

Substituting this into (20) yields

$$|\chi_\alpha(\sigma)| \le \sum_{\beta \trianglerighteq \alpha} |c_{\alpha\beta}| \cdot (n - \beta_1)! \cdot \xi_{\tilde{\beta}}(\sigma). \tag{21}$$

Substituting this bound into (19), we obtain

$$|w_\alpha| \le \sum_{\beta \trianglerighteq \alpha} |c_{\alpha\beta}| (n - \beta_1)! \cdot \frac{1}{n!} \sum_{\sigma \in A} \xi_{\tilde{\beta}}(\sigma).$$

We need the following lemma (which we will prove in Section 3).

Lemma 11. *Let $A \subset S_n$ be a family of permutations with $|A| = pn!$ and let $s \in \mathbb{N}$ with $s < t_p$. Then*

$$\tfrac{1}{n!} \sum_{\sigma \in A} \xi_{(n-s,s)}(\sigma) \le \frac{1}{K^{2K(1-\frac{7}{M})}}.$$

Now define $s(\beta) = n - \beta_1$. Notice that $\beta \trianglerighteq \alpha$ implies $s \le t$, so from (21) and Lemma 11 we obtain

$$\begin{aligned}
|w_\alpha| &\le \sum_{\beta \trianglerighteq \alpha} |c_{\alpha\beta}| \cdot s! \cdot \frac{1}{n!} \sum_{\sigma \in A} \xi_{(n-s,s)}(\sigma) \\
&\le \sum_{\beta \trianglerighteq \alpha} |c_{\alpha\beta}| \cdot s! \cdot \frac{1}{K^{2K(1-\frac{7}{M})}} \\
&\le \frac{t_p!}{K^{2K(1-\frac{7}{M})}} \cdot \sum_{\beta \trianglerighteq \alpha} |c_{\alpha\beta}|.
\end{aligned}$$

By Lemma 8, we have

$$\sum_{\beta \trianglerighteq \alpha} |c_{\alpha\beta}| \leq t_p!,$$

and therefore

$$|w_\alpha| \leq \frac{(t_p!)^2}{K^{2K(1-\frac{7}{M})}} \leq \frac{t_p^{2t_p}}{K^{2K(1-\frac{7}{M})}} \leq \frac{1}{K^{2K(1-\frac{8}{M})}},$$

proving Proposition 10. $\square$

Using (15), Corollary 7, Lemma 9 and Proposition 10, we obtain:

$$\begin{aligned}
|\partial A| &= n! \sum_{\alpha \vdash n} \mu_\alpha w_\alpha^2 \\
&\geq n! \sum_{\alpha_1 \leq n-t_p} \mu_\alpha w_\alpha^2 \\
&\geq n! \cdot \mu_{(n-t_p,t_p)} \cdot \sum_{\alpha_1 \leq n-t_p} w_\alpha^2 \\
&= n! \cdot (n \cdot t_p - (t_p)^2 + t_p) \cdot \left(\|1_A\|^2 - \sum_{\alpha_1 > n-t_p} w_\alpha^2 \right) \\
&\geq n! \cdot (n \cdot t_p - t_p^2) \cdot \left(p - |\{\alpha \vdash n : \alpha_1 > n - t_p\}| \cdot \left(\max_{\alpha_1 > n-t_p} |w_\alpha| \right)^2 \right) \\
&\geq n! \cdot (n \cdot t_p - t_p^2) \cdot \left(p - t_p! \cdot \left(\frac{1}{K^{2K(1-\frac{8}{M})}} \right)^2 \right) \\
&\geq n! \cdot (n \cdot t_p - t_p^2) \cdot \left(p - \frac{K^{\frac{K}{M}}}{K^{2K(2-\frac{16}{M})}} \right) \\
&\geq n! \cdot (n \cdot t_p - t_p^2) \cdot \left(p - \frac{1}{K^{2K(2-\frac{17}{M})}} \right) \\
&\geq n! \cdot (n \cdot t_p - t_p^2) \cdot (p - p^{2-\frac{17}{M}})
\end{aligned}$$

Taking $M = 18$, and using (18), we have $t_p \leq n/M = n/18$. Hence,

$$|\partial A| \geq n! \cdot (n \cdot t_p - t_p^2) \cdot p(1 - p^{1/18}) \geq c_0 \cdot t_p \cdot n \cdot |A|,$$

for some absolute constant $c_0 > 0$.

Using (18), it follows that

$$|\partial A| \geq c \cdot \frac{\log\left(\frac{1}{p}\right)}{\log\log\left(\frac{2}{p}\right)} \cdot n \cdot |A|$$

for some absolute constant $c > 0$, proving Theorem 3.

3 Proof of Lemma 11

Let $\sigma \in S_n$. For $i \in [n]$, let $C_i(\sigma)$ denote the number of cycles in σ of length i. Since $\xi_\alpha(\sigma)$ is the number of tabloids of shape α fixed by σ, it follows that $\xi_{(n-s,s)}(\sigma)$ is the number of subsets of $[n]$ of size s that are fixed by σ. Observe that if a set $S \subset [n]$ with $|S| = s$ is fixed by σ, then S is a union of at most s cycles of σ, all of which have length at most s. Therefore,

$$\xi_{(n-s,s)}(\sigma) \leq \left(\sum_{i=1}^{s} C_i(\sigma) \right)^s \leq s^{s-1} \cdot \sum_{i=1}^{s} (C_i(\sigma))^s, \tag{22}$$

using Jensen's inequality.

Hence,

$$\tfrac{1}{n!} \sum_{\sigma \in A} \xi_{(n-s,s)}(\sigma) \leq \frac{s^{s-1}}{n!} \sum_{i=1}^{s} \sum_{\sigma \in A} (C_i(\sigma))^s .$$

For $i \in [n]$ and $1 \leq j \leq \lfloor \frac{n}{i} \rfloor$, we define

$$D_{n,i,j} = \{\sigma \in S_n : C_i(\sigma) = j\}.$$

It was shown in [8] (cf. [2]) that

$$|D_{n,i,j}| = \frac{n! i^{-j}}{j!} \sum_{l=0}^{\lfloor n/i \rfloor - j} (-1)^l \frac{i^{-l}}{l!},$$

which implies

$$\tfrac{1}{3} \cdot \frac{n!}{i^j j!} \leq |D_{n,i,j}| \leq \frac{n!}{i^j j!}, \tag{23}$$

unless $i = 1$ and $j = n - 1$, in which case $|D_{n,i,j}| = 0$.

Fix a specific $i \in [s]$, and let $\kappa = \kappa_p(i) \in \mathbb{R}$ be such that

$$i^\kappa \cdot \kappa^\kappa = \frac{1}{p}. \tag{24}$$

Define $k = k_p(i) = \lfloor \kappa_p(i) \rfloor$. Clearly, for fixed p, $k_p(i)$ is monotone non-increasing in i, and therefore $k_p(i) \geq K - 1$ for all $1 \leq i \leq s$.

Define

$$D_{n,i,\geq k} := \bigcup_{j=k}^{\lfloor \frac{n}{i} \rfloor} D_{n,i,j}.$$

From the left-hand side of (23), it follows that

$$|D_{n,i,\geq k}| \geq |D_{n,i,k}| \geq \tfrac{1}{3} \frac{n!}{i^k k!} \geq \frac{n!}{i^k k^k} \geq \frac{n!}{i^\kappa \kappa^\kappa} = |A|, \tag{25}$$

using the fact that $n-2 \geq k \geq K-1 \geq 3$. Hence,

$$\frac{s^{s-1}}{n!}\sum_{\sigma\in A}(C_i(\sigma))^s \leq \frac{s^{s-1}}{n!}\left(\sum_{j=k}^{\lfloor \frac{n}{i}\rfloor}|D_{n,i,j}\cap A|\cdot j^s + |A\setminus D_{n,i,\geq k}|\cdot k^s\right).$$

Equation (25) implies that $|A\setminus D_{n,i,\geq k}| \leq |D_{n,i,\geq k}\setminus A|$. Hence,

$$\begin{aligned}\frac{s^{s-1}}{n!}\sum_{\sigma\in A}(C_i(\sigma))^s &\leq \frac{s^{s-1}}{n!}\left(\sum_{j=k}^{\lfloor \frac{n}{i}\rfloor}|D_{n,i,j}\cap A|\cdot j^s + |D_{n,i,\geq k}\setminus A|\cdot k^s\right)\\ &\leq \frac{s^{s-1}}{n!}\left(\sum_{j=k}^{\lfloor \frac{n}{i}\rfloor}|D_{n,i,j}|\cdot j^s\right).\end{aligned}$$

Using the right-hand inequality of (23), we obtain

$$\frac{s^{s-1}}{n!}\sum_{\sigma\in A}(C_i(\sigma))^s \leq s^{s-1}\sum_{j=k}^{\lfloor \frac{n}{i}\rfloor}\frac{j^s}{i^j j!} \leq \frac{s^{s-1}}{i^k}\sum_{j=k}^{\lfloor \frac{n}{i}\rfloor}\frac{j^s}{j!} \leq \frac{s^{s-1}}{i^k}\sum_{j=k}^{\infty}\frac{j^s}{j!}.$$

To bound the latter sum, we make the following claim.

Claim 2. *For any $s,k\in\mathbb{N}$ such that $Ms \leq k$ and $e^M \leq k$, we have*

$$\sum_{j=k}^{\infty}\frac{j^s}{j!} \leq \frac{1}{k^{(1-\frac{3}{M})k}}.$$

Proof of Claim 2. Using Stirling's bound $j! \geq (j/e)^j$ (valid for all $j\in\mathbb{N}$), we obtain

$$\sum_{j=k}^{\infty}\frac{j^s}{j!} \leq \sum_{j=k}^{\infty}\frac{j^s e^j}{j^j}.$$

The fact that $s \leq \frac{1}{M}j$ implies $j^s \leq j^{\frac{1}{M}j}$, whereas the fact that $2M < e^M \leq k \leq j$ implies that $j \geq 2M$ and $e^j \leq j^{\frac{j}{M}}$. Therefore,

$$\sum_{j=k}^{\infty}\frac{j^s e^j}{j^j} \leq \sum_{j=k}^{\infty}\frac{1}{j^{j-\frac{3}{M}j+2}} \leq \frac{1}{k^{(1-\frac{3}{M})k}}\sum_{j=k}^{\infty}\frac{1}{j^2} \leq \frac{1}{k^{(1-\frac{3}{M})k}},$$

proving the claim. $\square$

Coming back to the proof of Lemma 11, note from (17) that we have $k \geq \kappa - 1 \geq K-1 \geq e^M$ provided $p \leq \epsilon_0$ and ϵ_0 is a sufficiently small absolute constant. Hence, we may apply Claim 2, giving

$$\frac{s^{s-1}}{n!}\sum_{\sigma\in A}(C_i(\sigma))^s \leq \frac{s^{s-1}}{i^k}\sum_{j=k}^{\infty}\frac{j^s}{j!} \leq \frac{s^{s-1}}{i^k k^{(1-\frac{3}{M})k}} \leq \frac{s^{s-1}}{i^\kappa \kappa^{(1-\frac{6}{M})\kappa}},$$

where the last inequality follows from the fact that $k \geq \kappa - 1$. Equation (24) implies $i^\kappa \kappa^\kappa = K^{2K}$, which gives

$$\frac{s^{s-1}}{n!} \sum_{\sigma \in A} (C_i(\sigma))^s \leq \frac{s^{s-1}}{i^\kappa \kappa^{(1-\frac{6}{M})\kappa}} \leq \frac{s^{s-1}}{i^{(1-\frac{6}{M})\kappa} \kappa^{(1-\frac{6}{M})\kappa}} = \frac{s^{s-1}}{K^{2K(1-\frac{6}{M})}}.$$

Plugging this into equation (22) gives

$$\tfrac{1}{n!} \sum_{\sigma \in A} \xi_{(n-s,s)}(\sigma) \leq \frac{s^{s-1}}{n!} \sum_{i=1}^{s} \sum_{\sigma \in A} (C_i(\sigma))^s \leq \frac{s^s}{K^{2K(1-\frac{6}{M})}} \leq \frac{1}{K^{2K(1-\frac{7}{M})}},$$

proving Lemma 11.

4 Conclusion

For fixed each pair of positive integers n, k such that there exists a conjugation-invariant subset of S_n with size k, define

$$\Xi_n(k) = \min\{|\partial A| : \ A \subset S_n, \ A \text{ is conjugation invariant}, \ |A| = k\}.$$

We have given a lower bound on $\Xi_n(k)$ which is sharp up to an absolute constant factor. It would be interesting to determine more accurately the behaviour of $\Xi_n(k)$. We make the following conjecture in this regard.

Conjecture 12. *Let n, k be positive integers such that there exists a conjugation-invariant subset of S_n with size k. Let $s = s(n,k) \in \mathbb{N}$ be such that*

$$|A_s| \leq k \leq |A_{s-1}|,$$

where A_j is defined as in (3) for each $j \in \{0, 1, \ldots, n\}$. Then

$$\Xi_n(k) \geq \min\{|\partial(A_{s-1})|, |\partial(A_s)|\}.$$

At present, our methods do not seem capable of proving such an exact result.

5 Appendix

For completeness, we give here a proof of the bound (2) stated in the Introduction. First, we need a small amount of additional notation. If $i_1, \ldots, i_r \in [n]$ are distinct and $j_1, \ldots, j_r \in [n]$ are distinct, we write

$$R_{i_1 \mapsto j_1, i_2 \mapsto j_2, \ldots, i_r \mapsto j_r} := \{\sigma \in S_n : \ \sigma(i_k) = j_k \ \forall k \in [r]\}.$$

If $G = (V, E)$ is a finite graph and $S \subset V$, we write $G[S]$ for the subgraph of G induced on the set of vertices S, that is, the graph with vertex-set S, where vw is an edge of $G[S]$ if and only vw is an edge of G, for each $v, w \in S$. Moreover, if $S, T \subset V$ with $S \cap T = \emptyset$, we write $e(S, T)$ for the number of edges of G between S and T.

Proposition 13. *Let $t \in \{0, 1, \ldots, n-1\}$, and let $A \subset S_n$ be an initial segment of the lexicographic ordering on S_n, with $(n-t-1)! < |A| \leq (n-t)!$. Then*

$$|\partial A| \leq (t + 3/2)(n-1)|A|.$$

Proof. The case of general t will follow from the case $t = 0$, which we deal with in the following claim.

Claim 3. *Let $A \subset S_n$ be an initial segment of the lexicographic ordering on S_n with $|A| > (n-1)!$. Then $|\partial A| \leq \frac{3}{2}(n-1)|A|$.*

Proof of claim. By induction on n. The claim holds trivially for $n \leq 2$. Let $n \geq 3$, and assume the claim holds for $n-1$. Let $A \subset S_n$ be an initial segment of the lexicographic ordering on S_n with $|A| > (n-1)!$. Then we may write

$$A = R_{1 \mapsto 1} \cup R_{1 \mapsto 2} \cup \ldots \cup R_{1 \mapsto j-1} \cup A_1,$$

where $A_1 \subset R_{1 \mapsto j}$ and $j \in \{2, 3, \ldots, n\}$. Define

$$A_0 = R_{1 \mapsto 1} \cup R_{1 \mapsto 2} \cup \ldots \cup R_{1 \mapsto j-1};$$

then $A = A_0 \dot{\cup} A_1$. Define

$$C = R_{1 \mapsto j+1} \cup \ldots \cup R_{1 \mapsto n}.$$

Notice that

$$|\partial A| = e(A, C) + e(A_0, R_{1 \mapsto j} \setminus A_1) + e(A_1, R_{1 \mapsto j} \setminus A_1).$$

Observe that

$$e(A, C) + e(A_0, R_{1 \mapsto j} \setminus A_1) \leq (n-1)|A|,$$

since each edge on the left-hand side is between $T_{1 \mapsto i}$ and $T_{1 \mapsto i'}$ for some $i \neq i'$, and there are at most $n-1$ such edges of T_n incident with any permutation. To complete the proof of the inductive step, it suffices to show that

$$e(A_1, R_{1 \mapsto j} \setminus A_1) \leq \tfrac{1}{2}(n-1)|A|.$$

In fact, we prove the slightly stronger bound

$$e(A_1, R_{1 \mapsto j} \setminus A_1) \leq \tfrac{1}{2}(n-2)|A|,$$

by splitting into two cases.

Case (i): $\min\{|A_1|, |R_{1 \mapsto j} \setminus A_1|\} \leq (n-2)!$.

Notice that $e(A_1, R_{1 \mapsto j} \setminus A_1)$ is simply the size of the edge boundary of A_1 in the graph $T_n[R_{1 \mapsto j}]$, which is isomorphic to the transposition graph T_{n-1}. This graph is $\binom{n-1}{2}$-regular, so trivially,

$$e(A_1, R_{1 \mapsto j} \setminus A_1) \leq \binom{n-1}{2} \min\{|A_1|, |T_{1 \mapsto j} \setminus A_1|\}.$$

Since $\min\{|A_1|, |R_{1\mapsto j} \setminus A_1|\} \leq (n-2)! < |A|/(n-1)$, we have

$$e(A_1, R_{1\mapsto j} \setminus A_1) \leq \tfrac{1}{2}(n-2)|A|.$$

This completes the inductive step in case (i).

Case (ii): $\min\{|A_1|, |R_{1\mapsto j} \setminus A_1|\} > (n-2)!$.
Define

$$B = \begin{cases} A_1 & \text{if } |A_1| \leq \frac{1}{2}(n-1)!; \\ R_{1\mapsto j} \setminus A_1 & \text{otherwise.} \end{cases}$$

Then $|B| > (n-2)!$, and $e(B, R_{1\mapsto j} \setminus B)$ is the size of the edge-boundary of B in the graph $T_n[R_{1\mapsto j}]$. Hence, by the induction hypothesis, we have

$$e(B, R_{1\mapsto j} \setminus B) \leq \tfrac{3}{2}(n-2)|B|.$$

Since $|B| \leq \frac{1}{3}|A|$, we have

$$e(B, R_{1\mapsto j} \setminus B) \leq \tfrac{1}{2}(n-2)|A|.$$

Hence,

$$e(A_1, R_{1\mapsto j} \setminus A_1) \leq \tfrac{1}{2}(n-2)|A|.$$

This completes the inductive step in case (ii), proving the claim. □

We can now prove the proposition for $t \in [n-1]$. Let $t \in [n-1]$ and let $A \subset S_n$ be an initial segment of the lexicographic ordering on S_n with $(n-t-1)! < |A| \leq (n-t)!$. Then $A \subset R_{1\mapsto 1,\ldots,t\mapsto t}$. Hence,

$$|\partial A| = e(A, R_{1\mapsto 1,\ldots,t\mapsto t} \setminus A) + e(A, S_n \setminus R_{1\mapsto 1,\ldots,t\mapsto t}). \tag{26}$$

Observe that

$$e(A, S_n \setminus R_{1\mapsto 1,\ldots,t\mapsto t}) \leq t(n-1)|A|, \tag{27}$$

since each $\sigma \in R_{1\mapsto 1,\ldots,t\mapsto t}$ has exactly $t(n-1)$ neighbours in $S_n \setminus R_{1\mapsto 1,\ldots,t\mapsto t}$. Moreover, $e(A, R_{1\mapsto 1,\ldots,t\mapsto t} \setminus A)$ is simply the size of the edge boundary of A in the graph $T_n[R_{1\mapsto 1,\ldots,t\mapsto t}]$, which is isomorphic to the transposition graph on S_{n-t}. Hence, by Claim 3,

$$e(A, R_{1\mapsto 1,\ldots,t\mapsto t} \setminus A) \leq \tfrac{3}{2}(n-t-1)|A|. \tag{28}$$

Plugging (27) and (28) into (26) gives

$$|\partial A| \leq t(n-1)|A| + \tfrac{3}{2}(n-t-1)|A| \leq (t+3/2)(n-1)|A|.$$

This completes the proof of the proposition. □

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